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Time-independent canonical perturbation theory for nearly multiple-periodic systems

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Abstract. A modification of the classical canonical time-independent perturbation theory is presented for nearly multiple-periodic systems having Hamiltonians of the form

$$H = H_0(J_\alpha)+\lambda H_1(w_\alpha, J_\alpha; q_k, p_k)+\lambda^2 H_2(w_\alpha, J_\alpha; q_k, p_k)+\ldots$$

where $H_1, H_2, \ldots$ are periodic functions of the angles w_α. The perturbation procedure is based on averaging of the Hamilton–Jacobi equation over the angles w_α. The existence of constants of motion to all orders of the perturbation theory for both non-degenerate and intrinsically degenerate systems is shown.

1. Introduction

The Poincaré canonical time-independent perturbation theory is based on an approximate solution of the Hamilton–Jacobi equation transformed to angle-action variables (Born 1960, Fues 1927). By its very nature, this method is adapted for calculation of multiple-periodic motions of conservative systems and fails if the unperturbed motion is not multiple-periodic, or if its multiple-periodicity is destroyed by the perturbation. Since, in plasma physics for example, one steadily encounters infinite motions due to linear as well as nonlinear instabilities, it is of interest to find a perturbation procedure which would be applicable also to such systems. Thus, Coffey (1969), using the Mitropolskii–Zubarev method of rapidly rotating phase (Bogolyubov and Mitropolskii 1955) generalized by Coffey and Ford (1969), treated nearly multiple-periodic systems having Hamiltonians of the form

$$H = H_0(J^0_\alpha)+\lambda H_1(w^0_\alpha, J^0_\alpha; q^0_k, p^0_k) \tag{1.1}$$

where H_1 is a periodic function of the angles w^0_α. Coffey succeeded in showing that, for a non-degenerate system, to each angle w^0_α there corresponds an averaged quantity J_α (the new action) which is constant to all orders of the perturbation theory. In the degenerate case such a quantity can be associated only with the proper angles (to be defined later), provided that the average variables can be made canonical. The conditions for this, however, are not known.

In this paper another approach to nearly multiple-periodic systems is presented which by-passes these difficulties. We do not seek an averaged Hamiltonian as Coffey did, but, rather, an averaged Hamilton–Jacobi equation for the generator (the Hamilton characteristic function) of the canonical transformation that converts the original Hamiltonian into one containing only momenta. In § 2 non-degenerate systems are treated. A suitable *ansatz* for the Hamilton characteristic function makes it possible to solve the Hamilton–Jacobi equation by perturbation theory in such a way that from the requirement that the perturbed Hamilton characteristic function be periodic in the angles w_α^0 a reduced Hamilton–Jacobi equation results which involves only the variables corresponding to the non-periodic degrees of freedom. If these are not too many, the resulting equation may be tractable. This is certainly so if there is only one non-periodic degree of freedom.

In § 3 Hamiltonians of intrinsically degenerate nearly multiple-periodic systems are shown to be reducible to the form treated in § 2. Corresponding to the resonance relations between the unperturbed frequencies, the transformation to improper angle-variables is performed so that the unperturbed Hamiltonian does not contain the corresponding actions. The improper angle-action variables then occur only in the perturbing Hamiltonians and are treated in the same way as the non-periodic variables q_k^0, p_k^0. In comparison with the usual degenerate perturbation theory (Born 1960, Fues 1927) this procedure is not only conceptually simpler, but also enables one to treat systems in which the perturbation incites an instability. Moreover, the existence of constants of motion to all orders of the perturbation theory may be shown in exactly the same way in both the non-degenerate and the degenerate case without any restrictions.

In § 4, as an application of the general theory, a degenerate Hamiltonian system of three harmonic oscillators coupled by nonlinear forces is examined.

2. Non-degenerate systems

We shall deal with nearly multiple-periodic systems having Hamiltonians of the form

$$H(w_\alpha^0, J_\alpha^0; q_k^0, p_k^0; \lambda) = H_0(J_\alpha^0) + \lambda H_1(w_\alpha^0, J_\alpha^0; q_k^0, p_k^0) + \lambda^2 H_2(w_\alpha^0, J_\alpha^0; q_k^0, p_k^0) + \dots \tag{2.1}$$

in which w_α^0, J_α^0 $(\alpha = 1, 2, \dots, m)$ are the angle-action variables corresponding to the unperturbed motion which is thus assumed to be multiple-periodic, and

$$q_k^0, \qquad p_k^0 \qquad (k = 1, 2, \dots, n)$$

are the canonically conjugate variables of the non-periodic degrees of freedom. The unperturbed motion is assumed to be non-degenerate in the sense that the unperturbed frequencies

$$\Omega_\beta^0(J_\alpha^0) = \frac{\partial H_0(J_\alpha^0)}{\partial J_\beta^0} \tag{2.2}$$

are all nonzero and rationally independent. λ is a small parameter and the functions $H_1, H_2, \dots$ are periodic in the angles w_α^0 with period 2π.

The Hamilton characteristic function S, generating the transformation of variables such that the new Hamiltonian is a function of only the new momenta, is expanded as usual into a power series in λ, the lowest-order term being an identity transformation of the angle-action variables w_α^0, J_α^0 plus an unknown transformation T of the variables q_k^0, p_k^0 parametrically dependent on the new actions J_α to take account of the coupling between the periodic and non-periodic modes of motion (the new variables are without the superscript zero)

$$S(w_\alpha^0, J_\alpha; q_k^0, p_k)$$

$$= \sum_{\alpha=1}^{m} w_\alpha^0 J_\alpha + T(J_\alpha, q_k^0, p_k) + \lambda \mathscr{S}_1(w_\alpha^0, J_\alpha; q_k^0, p_k) + \lambda^2 \mathscr{S}_2(w_\alpha^0, J_\alpha; q_k^0, p_k) + \dots \tag{2.3}$$

The transformation relations generated by S,

$$w_\alpha = w_\alpha^0 + \frac{\partial T}{\partial J_\alpha} + \lambda \frac{\partial \mathscr{S}_1}{\partial J_\alpha} + \lambda^2 \frac{\partial \mathscr{S}_2}{\partial J_\alpha} + \dots \tag{2.4}$$

$$J_\alpha^0 = J_\alpha + \lambda \frac{\partial \mathscr{S}_1}{\partial w_\alpha^0} + \lambda^2 \frac{\partial \mathscr{S}_2}{\partial w_\alpha^0} + \dots \tag{2.5}$$

$$q_k = \frac{\partial T}{\partial p_k} + \lambda \frac{\partial \mathscr{S}_1}{\partial p_k} + \lambda^2 \frac{\partial \mathscr{S}_2}{\partial p_k} + \dots \tag{2.6}$$

$$p_k^0 = \frac{\partial T}{\partial q_k^0} + \lambda \frac{\partial \mathscr{S}_1}{\partial q_k^0} + \lambda^2 \frac{\partial \mathscr{S}_2}{\partial q_k^0} + \dots \tag{2.7}$$

are now used to eliminate J_α^0 and p_k^0 from the Hamiltonian (2.1). Choosing the new Hamiltonian in the form $E + \lambda \mathscr{E}$ we thus obtain the Hamilton–Jacobi equation for S. This equation is expanded into powers of λ and on putting E equal to $H_0(J_\alpha)$ there results

$$\sum_{\alpha=1}^{m} \frac{\partial H_0}{\partial J_\alpha} \frac{\partial \mathscr{S}_1}{\partial w_\alpha^0} + H_1\left(w_\alpha^0, J_\alpha; q_k^0, \frac{\partial T}{\partial q_k^0}\right) + \lambda \left[\sum_{\alpha=1}^{m} \frac{\partial H_0}{\partial J_\alpha} \frac{\partial \mathscr{S}_2}{\partial w_\alpha^0} + \frac{1}{2} \sum_{\alpha_1=1}^{m} \sum_{\alpha_2=1}^{m} \frac{\partial^2 H_0}{\partial J_{\alpha_1} \partial J_{\alpha_2}} \frac{\partial \mathscr{S}_1}{\partial w_{\alpha_1}^0} \frac{\partial \mathscr{S}_1}{\partial w_{\alpha_2}^0} \right.$$
$$\left. + \sum_{\alpha=1}^{m} \frac{\partial H_1}{\partial J_\alpha} \frac{\partial \mathscr{S}_1}{\partial w_\alpha^0} + \sum_{k=1}^{n} \frac{\partial H_1}{\partial p_k^0} \bigg| \frac{\partial \mathscr{S}_1}{\partial q_k^0} + H_2\left(w_\alpha^0, J_\alpha; q_k^0, \frac{\partial T}{\partial q_k^0}\right) \right] + \dots = \mathscr{E} \tag{2.8}$$

(the vertical line is to denote that, on differentiation, $\partial T / \partial q_k^0$ is to be substituted for p_k^0). To extract from this relation the equations for $\mathscr{S}_1, \mathscr{S}_2, \dots$ and T, an averaging process with respect to the angles w_α^0 will be performed. A periodic function $F(w_\alpha^0)$ is decomposed into the mean (averaged) value $\langle F(w_\alpha^0) \rangle$ defined by

$$\langle F(w_\alpha^0) \rangle = \iint \cdots \int_0^{2\pi} F(w_\alpha^0)\, \mathrm{d}w_1^0\, \mathrm{d}w_2^0 \dots \mathrm{d}w_m^0 \tag{2.9}$$

and the periodic part

$$\{F(w_\alpha^0)\} = F(w_\alpha^0) - \langle F(w_\alpha^0) \rangle. \tag{2.10}$$

The functions $\mathscr{S}_1, \mathscr{S}_2, \dots$ are determined so as to be periodic functions of the angles

$$\sum_{\alpha=1}^{m} \frac{\partial H_0}{\partial J_\alpha} \frac{\partial \mathscr{S}_1}{\partial w_\alpha^0} + \left\{ H_1\left(w_\alpha^0, J_\alpha; q_k^0, \frac{\partial T}{\partial q_k^0}\right) \right\} = 0 \tag{2.11a}$$

$$\sum_{\alpha=1}^{m} \frac{\partial H_0}{\partial J_\alpha} \frac{\partial \mathscr{S}_2}{\partial w_\alpha^0} + \left\{ \frac{1}{2} \sum_{\alpha_1=1}^{m} \sum_{\alpha_2=1}^{m} \frac{\partial^2 H_0}{\partial J_{\alpha_1} \partial J_{\alpha_2}} \frac{\partial \mathscr{S}_1}{\partial w_{\alpha_1}^0} \frac{\partial \mathscr{S}_1}{\partial w_{\alpha_2}^0} + \sum_{\alpha=1}^{m} \frac{\partial H_1}{\partial J_\alpha} \frac{\partial \mathscr{S}_1}{\partial w_\alpha^0} + \sum_{k=1}^{n} \frac{\partial H_1}{\partial p_k^0} \bigg| \frac{\partial \mathscr{S}_1}{\partial q_k^0} \right.$$
$$\left. + H_2\left(w_\alpha^0, J_\alpha; q_k^0, \frac{\partial T}{\partial q_k^0}\right) \right\} = 0. \tag{2.11b}$$

$$\vdots$$

To suppress secularities, the function T is defined so as to nullify the averaged part of equation (2.8)

$$\left\langle H_1\left(w_\alpha^0, J_\alpha; q_k^0, \frac{\partial T}{\partial q_k^0}\right)\right\rangle + \lambda\left\langle \frac{1}{2}\sum_{\alpha_1=1}^{m}\sum_{\alpha_2=1}^{m} \frac{\partial^2 H_0}{\partial J_{\alpha_1}\partial J_{\alpha_2}} \frac{\partial \mathscr{S}_1}{\partial w_{\alpha_1}^0} \frac{\partial \mathscr{S}_1}{\partial w_{\alpha_2}^0} + \sum_{\alpha=1}^{m} \frac{\partial H_1}{\partial J_\alpha}\frac{\partial \mathscr{S}_1}{\partial w_\alpha^0} + \sum_{k=1}^{n} \frac{\partial H_1}{\partial p_k^0}\left|\frac{\partial \mathscr{S}_1}{\partial q_k^0}\right. + H_2\left(w_\alpha^0, J_\alpha; q_k^0, \frac{\partial T}{\partial q_k^0}\right)\right\rangle + \ldots = \mathscr{E}. \tag{2.12}$$

Thus the calculational procedure is as follows: we first calculate the functions $\mathscr{S}_1, \mathscr{S}_2, \ldots$ from equations (2.11*a*), (2.11*b*), ... in the form of multiple Fourier series of the angles w_α^0 the coefficients of which depend on the new actions J_α, the old coordinates q_k^0 and the derivatives of the function T with respect to q_k^0. The functions $\mathscr{S}_1, \mathscr{S}_2, \ldots$ are then substituted into equation (2.12) so that, on averaging, a reduced Hamilton–Jacobi equation for the function T results

$$G\left(J_\alpha; q_k^0, \frac{\partial T}{\partial q_k^0}\right) = \mathscr{E}. \tag{2.12}$$

The new actions J_α occur in this equation as mere parameters. A complete solution of this equation is to be sought containing n arbitrary constants p_k (in addition to the arbitrary constants J_α) one of which (eg p_1) is taken to be $\mathscr{E}$. Any other set of n independent functions P_k of these constants

$$P_k = P_k(J_\alpha; \mathscr{E}, p_2, p_3, \ldots, p_n) \tag{2.13}$$

may be taken to be the set of new momenta. By solving these relations we may express $\mathscr{E}, p_2, p_3, \ldots, p_n$ as functions of J_α, P_k

$$\begin{aligned} \mathscr{E} &= \mathscr{E}(J_\alpha, P_k) \\ p_2 &= p_2(J_\alpha, P_k) \\ &\vdots \\ p_n &= p_n(J_\alpha, P_k) \end{aligned} \tag{2.14}$$

and, accordingly, regard the function S (2.3) as a generator of the transformation from q_k^0, p_k^0 to Q_k, P_k:

$$\begin{aligned} S(w_\alpha^0, J_\alpha; q_k^0, p_k(J_\alpha, P_k)) &= \sum_{\alpha=1}^{m} w_\alpha^0 J_\alpha + T(J_\alpha; q_k^0, p_k(J_\alpha, P_k)) + \lambda\mathscr{S}_1(w_\alpha^0, J_\alpha; q_k^0, p_k(J_\alpha, P_k)) \\ &\quad + \lambda^2\mathscr{S}_2(w_\alpha^0, J_\alpha; q_k^0, p_k(J_\alpha, P_k)) + \ldots \\ &= \sum_{\alpha=1}^{m} w_\alpha^0 J_\alpha + T'(J_\alpha; q_k^0, P_k) + \lambda\mathscr{S}'_1(w_\alpha^0, J_\alpha; q_k^0, P_k) + \lambda^2\mathscr{S}'_2(w_\alpha^0, J_\alpha; q_k^0, P_k) + \ldots. \end{aligned} \tag{2.15}$$

Instead of the transformation relations (2.4) to (2.7) we then have

$$w_\alpha = w_\alpha^0 + \frac{\partial T'}{\partial J_\alpha} + \lambda\frac{\partial \mathscr{S}'_1}{\partial J_\alpha} + \lambda^2\frac{\partial \mathscr{S}'_2}{\partial J_\alpha} + \ldots \tag{2.16}$$

$$J_\alpha^0 = J_\alpha + \lambda\frac{\partial \mathscr{S}'_1}{\partial w_\alpha^0} + \lambda^2\frac{\partial \mathscr{S}'_2}{\partial w_\alpha^0} + \ldots \tag{2.17}$$

$$Q_k = \frac{\partial T'}{\partial P_k} + \lambda \frac{\partial \mathscr{S}'_1}{\partial P_k} + \lambda^2 \frac{\partial \mathscr{S}'_2}{\partial P_k} + \dots \tag{2.18}$$

$$p_k^0 = \frac{\partial T'}{\partial q_k^0} + \lambda \frac{\partial \mathscr{S}'_1}{\partial q_k^0} + \lambda^2 \frac{\partial \mathscr{S}'_2}{\partial q_k^0} + \dots . \tag{2.19}$$

It may happen that the perturbed motion is again multiple-periodic. This becomes manifest if the averaged Hamilton–Jacobi equation (2.12) is separable so that its solution may be written in the form

$$T(J_\alpha; q_k^0, p_k) = T_1(J_\alpha; q_1^0, p_k) + T_2(J_\alpha; q_2^0, p_k) + \dots + T_n(J_\alpha; q_n^0, p_k) \tag{2.20}$$

and the phase-plane trajectories representing the dependence of $\partial T/\partial q_k^0$ on q_k^0 are either closed (libration) or periodic (rotation). Then the actions J_k may be introduced by the integrals

$$J_k = \frac{1}{2\pi} \oint \frac{\partial T_k}{\partial q_k^0} \, \mathrm{d}q_k^0. \tag{2.21}$$

This is a special case of the relations (2.13). All degrees of freedom of the perturbed system are then described by angle-action variables $w_\alpha, J_\alpha; w_k, J_k$. The separability of the Hamilton–Jacobi equation (2.12) is of course not necessary in order that the angle-action variables may be introduced.

The solution of the problem is completed by solving the equations of motion obeyed by the new variables w_α, J_α; Q_k, P_k

$$\dot{w}_\alpha = \frac{\partial E(J_\alpha)}{\partial J_\alpha} + \lambda \frac{\partial \mathscr{E}(J_\alpha, P_k)}{\partial J_\alpha} = \Omega_\alpha^0(J_\alpha) + \lambda \frac{\partial \mathscr{E}(J_\alpha, P_k)}{\partial J_\alpha} \tag{2.22}$$

$$\dot{J}_\alpha = 0 \tag{2.23}$$

$$\dot{Q}_k = \lambda \frac{\partial \mathscr{E}(J_\alpha, P_k)}{\partial P_k} \tag{2.24}$$

$$\dot{P}_k = 0. \tag{2.25}$$

All the new actions J_α are therefore constants to all orders of the perturbation theory.

3. Degenerate systems

We again assume that the Hamiltonian has the form (2.1)

$$\begin{aligned} &H(w_\mu^{(0)}, J_\mu^{(0)}; q_k^{(0)}, p_k^{(0)}; \lambda) \\ &\quad = H_0(J_\mu^{(0)}) + \lambda H_1(w_\mu^{(0)}, J_\mu^{(0)}; q_k^{(0)}, p_k^{(0)}) + \lambda^2 H_2(w_\mu^{(0)}, J_\mu^{(0)}; q_k^{(0)}, p_k^{(0)}) + \dots \\ &\quad \mu = 1, 2, \dots, m; \qquad k = 1, 2, \dots, n \end{aligned} \tag{3.1}$$

but this time the unperturbed frequencies (2.2) fulfil s resonance relations

$$\begin{aligned} &l_{\rho 1}\Omega_1^{(0)} + l_{\rho 2}\Omega_2^{(0)} + \dots + l_{\rho m}\Omega_m^{(0)} = 0 \\ &\rho = m-s+1, m-s+2, \dots, m \end{aligned} \tag{3.2}$$

where l_{ij} are integers and not all $l_{\rho 1}, l_{\rho 2}, \dots, l_{\rho n}$ vanish. Corresponding to these relations we introduce improper angles w_ρ^0 by a homogeneous linear transformation with coefficients $l_{\rho 1}$ (Born 1960, Fues 1927)

$$w_\rho^0 = l_{\rho 1} w_1^{(0)} + l_{\rho 2} w_2^{(0)} + \ldots + l_{\rho m} w_m^{(0)} \tag{3.3}$$

leaving the remaining (proper) angles $w_\alpha^{(0)}$ ($\alpha = 1, 2, \ldots, m-s$) and the variables $q_k^{(0)}$, $p_k^{(0)}$ ($k = 1, 2, \ldots, n$) unchanged. This transformation is generated by the function

$$S(w_\mu^{(0)}, J_\mu^0; q_k^{(0)}, p_k^0) = \sum_{\alpha=1}^{m-s} w_\alpha^{(0)} J_\alpha^0 + \sum_{\rho=m-s+1}^{m} (l_{\rho 1} w_1^{(0)} + l_{\rho 2} w_2^{(0)} + \ldots + l_{\rho m} w_m^{(0)}) J_\rho^0 + \sum_{k=1}^{n} q_k^{(0)} p_k^0. \tag{3.4}$$

We assume that the degeneracy is intrinsic so that the new unperturbed Hamiltonian $H_0'(J_\alpha^0)$ does not depend on the improper actions J_ρ^0. Thus, in terms of the new variables introduced by the transformation (3.4), the new Hamiltonian becomes

$$H'(w_\alpha^0, J_\alpha^0; w_\rho^0, J_\rho^0; q_k^0, p_k^0; \lambda) = H_0'(J_\alpha^0) + \lambda H_1'(w_\alpha^0, J_\alpha^0; w_\rho^0, J_\rho^0; q_k^0, p_k^0) + \lambda^2 H_2'(w_\alpha^0, J_\alpha^0; w_\rho^0, J_\rho^0; q_k^0, p_k^0) + \ldots . \tag{3.5}$$

In this way we have in fact reduced the problem to the one treated in the foregoing section. To see this it is enough to put

$$q_{k+r}^0 = w_{m-s+r}^0; \qquad p_{k+r}^0 = w_{m-s+r}^0 \qquad r = 1, 2, \ldots, s \tag{3.6}$$

and to rewrite the Hamiltonian (3.5) accordingly. We obtain the Hamiltonian (2.1) with the only difference that the number of pairs of the angle-action variables w_α^0, J_α^0 is diminished to $m-s$ and the number of pairs of the non-periodic variables q_k^0, p_k^0 increased to $n+s$. The perturbation procedure is then the same as described previously.

Only the actions J_α corresponding to the proper angles are constants of the motion to all orders of the perturbation theory. This, as follows from equation (2.23), is true without any restrictions.

4. Example

As an example illustrating the general theory we shall examine a degenerate Hamiltonian system of three harmonic oscillators coupled by nonlinear forces in such a way that the equilibrium is, in the case where the resonance relation is exactly fulfilled, unstable. This phenomenon occurs in the theory of nonlinear interaction of electromagnetic waves with positive and negative energy in non-thermal plasmas where it is known under the name of explosive instability. For the physical background of the problem the reader is referred to Sedláček (1974).

The Hamiltonian is assumed in the form (3.1), with variables $q_k^{(0)}$, $p_k^{(0)}$ missing

$$H(w_1^{(0)}, J_1^{(0)}; w_2^{(0)}, J_2^{(0)}; w_3^{(0)}, J_3^{(0)}; \lambda) = H_0(J_1^{(0)}, J_2^{(0)}, J_3^{(0)}) + \lambda H_1(w_1^{(0)}, J_1^{(0)}; w_2^{(0)}, J_2^{(0)}; w_3^{(0)}, J_3^{(0)}) \tag{4.1}$$

$$H_0 = \Omega_1^0 J_1^{(0)} + \Omega_2^0 J_2^{(0)} + \Omega_3^0 J_3^{(0)} \tag{4.2}$$

$$H_1 = K(J_1^{(0)}J_2^{(0)}J_3^{(0)})^{1/2}\cos(w_1^{(0)}+w_2^{(0)}+w_3^{(0)})+K_1J_1^{(0)}\sqrt{J_3^{(0)}}\cos(2w_1^{(0)}+w_3^{(0)})$$
$$+K_2J_2^{(0)}\sqrt{J_3^{(0)}}\cos(2w_2^{(0)}+w_3^{(0)}). \qquad (4.3)$$

Thus H_0 represents the system of three harmonic oscillators in angle-action variables and H_1 their nonlinear coupling (K, K_1, K_2 are the coupling constants) which is regarded as perturbation. The formal expansion parameter λ is put equal to one. The unperturbed frequencies $\Omega_1^0, \Omega_2^0, \Omega_3^0$ fulfil one resonance relation (3.2)

$$\Omega_1^0+\Omega_2^0+\Omega_3^0 = 0 \qquad (4.4)$$

so that the canonical transformation introducing one improper angle w_3^0 and its conjugate action J_3^0 reads

$$w_1^{(0)} = w_1^0, \qquad w_2^{(0)} = w_2^0, \qquad w_3^{(0)} = w_3^0-w_1^0-w_2^0 \qquad (4.5)$$

$$J_1^{(0)} = J_1^0+J_3^0, \qquad J_2^{(0)} = J_2^0+J_3^0, \qquad J_3^{(0)} = J_3^0. \qquad (4.6)$$

If, according to (3.6), the notation for the pair of variables w_3^0, J_3^0 is changed to q_1^0, p_1^0, the transformed Hamiltonian takes on the form (2.1)

$$H'(w_1^0, J_1^0; w_2^0, J_2^0; q_1^0, p_1^0) = H_0'(J_1^0, J_2^0)+\lambda H_1'(w_1^0, J_1^0; w_2^0, J_2^0; q_1^0, p_1^0) \qquad (4.7)$$

$$H_0' = \Omega_1^0J_1^0+\Omega_2^0J_2^0 \qquad (4.8)$$

$$H_1' = K[(J_1^0+p_1^0)(J_2^0+p_1^0)p_1^0]^{1/2}\cos q_1^0+K_1(J_1^0+p_1^0)\sqrt{p_1^0}\cos(w_1^0-w_2^0+q_1^0)$$
$$+K_2(J_2^0+p_1^0)\sqrt{p_1^0}\cos(-w_1^0+w_2^0+q_1^0). \qquad (4.9)$$

The generating function (2.3) is now assumed in the form

$$S(w_1^0, J_1; w_2^0, J_2; q_1^0, p_1^0)$$
$$= w_1^0J_1+w_2^0J_2+T(J_1, J_2; q_1^0, p_1)+\lambda\mathscr{S}_1(w_1^0, J_1; w_2^0, J_2; q_1^0, p_1)+\dots. \qquad (4.10)$$

Restricting ourselves to the second-order approximation to the averaged Hamilton–Jacobi equation (2.12), it is sufficient to calculate only the function $\mathscr{S}_1$ from equation (2.11*a*)

$$\Omega_1^0\frac{\partial\mathscr{S}_1}{\partial w_1^0}+\Omega_2^0\frac{\partial\mathscr{S}_1}{\partial w_2^0}$$
$$= K_1(J_1+p_1)\sqrt{p_1}\cos(w_1^0-w_2^0+q_1^0)+K_2(J_2+p_1)\sqrt{p_1}\cos(-w_1^0+w_2^0+q_1^0). \qquad (4.11)$$

The solution is easily found

$$\mathscr{S}_1 = -(\Omega_1^0-\Omega_2^0)^{-1}K_1(J_1+p_1)\sqrt{p_1}\sin(w_1^0-w_2^0+q_1^0)$$
$$+(\Omega_1^0-\Omega_2^0)^{-1}K_2(J_2+p_1)\sqrt{p_1}\sin(-w_1^0+w_2^0+q_1^0). \qquad (4.12)$$

On substitution into equation (2.12) one obtains the second-order Hamilton–Jacobi equation, averaged over the proper angles w_1^0, w_2^0, from which the function T is to be determined

$$K\left[\left(J_1+\frac{\partial T}{\partial q_1^0}\right)\left(J_2+\frac{\partial T}{\partial q_1^0}\right)\frac{\partial T}{\partial q_1^0}\right]^{1/2}\cos q_1^0$$
$$-\lambda\left[\frac{1}{4}\frac{K_1^2J_1^2-K_2^2J_2^2}{\Omega_1^0-\Omega_2^0}+\frac{3}{2}\frac{K_1^2J_1-K_2^2J_2}{\Omega_1^0-\Omega_2^0}\frac{\partial T}{\partial q_1^0}+\frac{5}{4}\frac{K_1^2-K_2^2}{\Omega_1^0-\Omega_2^0}\left(\frac{\partial T}{\partial q_1^0}\right)^2\right] = \mathscr{E}. \qquad (4.13)$$

The new actions J_1, J_2 are constants of the motion; they occur in equation (4.13) as mere parameters. Equation (4.13) is therefore the Hamilton–Jacobi equation of a dynamical system with only one degree of freedom, corresponding to the variables q_1, p_1. To solve this equation, which is an ordinary differential equation of the first-order, is certainly simpler than to solve the original set of six first-order canonical equations generated by the Hamiltonian (4.1). However, to obtain an overall picture of the motion, solution of the Hamilton–Jacobi equation (4.13) need not be attempted because for this purpose the examination of this equation in the phase-plane $(q_1^0, \partial T/\partial q_1^0)$ is sufficient. Thus one finds that the first-order approximation to the averaged Hamilton–Jacobi equation† (the first term on the left-hand side of equation (4.13)) generates only infinite non-periodic motions and that the equilibrium point $J_1 = J_2 = p_1 = 0$ is unstable. The usual degenerate perturbation theory would be inapplicable under such circumstances. The second-order approximation to the averaged Hamilton–Jacobi equation (the complete equation (4.13)) on the other hand limits the motion to a finite region of the phase-plane though the equilibrium point remains unstable. For further discussion which includes also the case when the resonance relation (4.4) is not fulfilled exactly, the reader is again referred to the paper of Sedláček (1974).

The new Hamiltonian

$$H_0^{\mathrm{T}}(J_1, J_2; q_1, p_1) = H_0(J_1, J_2) + \lambda\mathscr{E} \qquad (4.14)$$

generates, if $\mathscr{E}$ is regarded as a constant independent of J_1, J_2, p_1 (which is one of the possible choices of the function (2.14)), the following canonical equations of motion

$$\dot{w}_1 = \Omega_1^0, \qquad \dot{J}_1 = 0 \qquad (4.15)$$

$$\dot{w}_2 = \Omega_2^0, \qquad \dot{J}_2 = 0 \qquad (4.16)$$

$$\dot{q}_1 = 0, \qquad \dot{p}_1 = 0. \qquad (4.17)$$

The solution of these equations, transformed back to the original variables $w_1^{(0)}$, $J_1^{(0)}$, $w_2^{(0)}$, $J_2^{(0)}$, $w_3^{(0)}$, $J_3^{(0)}$ then gives the solution of the original canonical equations associated with the Hamiltonian (4.1).

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† This equation is identical with the so called equation for secular motions of the usual degenerate perturbation theory (Born 1960, Fues 1927).